Problem 1. For positive integers $a_1, \dots, a_k$, let $n = \sum_{i=1}^{k} a_i$, and let $\binom{n}{a_1, \dots, a_k}$ be the multinomial coefficient $\dfrac{n!}{\prod_{i=1}^{k}(a_i!)}$.

Let $d = \gcd(a_1, \dots, a_k)$ denote the greatest common divisor of $a_1, \dots, a_k$. Prove that $\dfrac{d}{n}\binom{n}{a_1, \dots, a_k}$ is an integer.

Problem 2. A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z), where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\big((n+2)\sqrt{n/3},\, n\sqrt{6}\big)$.

Problem 3. Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcenter of $\Delta A_jA_kA_\ell$, with $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

Assuming $A_i \ne O_i$ for all indices i, prove that the four lines A_iO_i are concurrent or parallel.

Problem 4. For a finite set X of positive integers, let

$$\Sigma(X) = \sum_{x \in X} \arctan\frac{1}{x}.$$

Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which

$$S \subset T \text{ and } \Sigma(T) = \frac{\pi}{2}.$$

Every problem is worth 7 points.
Time allowed is 5 hours.